# COMPUTATION OF TURBULENT BOUNDARY LAYER USING SIMILARITY SOLUTION

V. V. Zyabrikov

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1. Formulation of Universal Turbulent Boundary-Layer Equations. An arbitrary choice of the representative family of skin friction or velocity profiles is used in the integral methods of turbulent boundary-layer computations. If a two-layer semiempirical scheme is used in computations, then it becomes necessary to carry out the laborious process of matching the inner and outer solutions in each case. The method of similar solutions makes it possible to complete the matching process once and for all.

Reynolds' equation for plane, incompressible, turbulent flow is used in the form

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=U\frac{dU}{dx}+\frac{1}{\rho}\frac{\partial\tau}{\partial y}, \tag{1.1}$$

$$\psi=0,\quad \frac{\partial\psi}{\partial y}=0 \quad \text{at} \quad y=0,\quad \frac{\partial\psi}{\partial y}\to U(x) \quad \text{as} \quad y\to\infty,$$

$$\frac{\partial\psi}{\partial y}=u_0(y) \quad \text{at} \quad x=x_0,$$

where $\psi$ is the stream function for the mean flow; U, external flow velocity; $\tau$, total shear stress (sum of viscous and eddy stresses); x, y, longitudinal and transverse coordinates, respectively; $u_0(y)$, starting profile for the longitudinal velocity at the chosen initial point $x=x_0$; $\rho$, fluid density.

We choose U(x) as the reference velocity for longitudinal velocities at different sections of the boundary layer and the transverse length scale is the momentum thickness $\delta^{**}$. It follows from the expansion of the function $\tau(x, y)$ in terms of the transverse coordinate y

$$\tau=\tau_w+\left.\frac{\partial\tau}{\partial y}\right|_{y=0}y+o(y^3)=\tau_w-\rho U\frac{dU}{dx}y+o(y^3),$$

that the usual shear stress reference $\tau_W$ can be replaced by the quantity $\tau_w-\rho U\delta^{**}dU/dx$, which takes into consideration the presence of streamwise pressure gradient and does not become zero at the separation point. We choose the same set of similarity parameters as in the case of laminar boundary layer [1]:

$$f_k=U^{k-1}\frac{d^kU}{dx^k}z^k \quad (k=1,2,3\ldots),$$

where $z=\delta^{**}/(Uc)$; $c=c_f/2-(dU/dx)\delta^{**}/U$.

Equation (1.1) is subjected to the similarity transformation

$$\eta=y/\delta^{**},\ \Phi=\psi/(U\delta^{**})=\Phi[\eta,\ \mathrm{Re}^{**},\ (f_k)],$$

$$\zeta=\tau\Big/\left(\tau_w-\rho U\frac{dU}{dx}\delta^{**}\right)=\zeta[\eta,\ \mathrm{Re}^{**},\ (f_k)],$$

which, unlike the laminar boundary layer, contains "local" Reynolds number $\mathrm{Re}^{**}=U\delta^{**}/\nu$. $\mathrm{Re}^{**}$ will henceforth be considered a parameter (the possibility of such an approach was shown in [2]).

Equations for the determination of $df_k/dx$ coincide with the corresponding laminar boundary-layer equations

$$Uz\frac{df_k}{dx}=[(k-1)f_1+k\bar{f}_1]f_k+f_{k+1}\equiv\theta_k. \tag{1.2}$$

The only significant difference is the more complex expression

$$\bar{f}_1=\frac{1-(2+H)f_1-\sum\limits_{k=1}^{\infty}[(k-1)f_1f_k+f_{k+1}]\dfrac{\partial\ln c}{\partial f_k}}{1+\sum\limits_{k=1}^{\infty}kf_k\dfrac{\partial\ln c}{\partial f_k}}, \tag{1.3}$$

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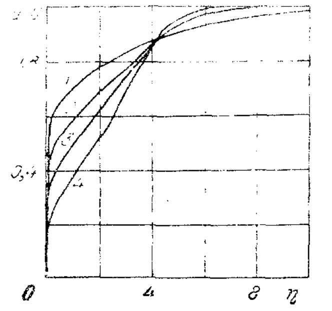


Fig. 1

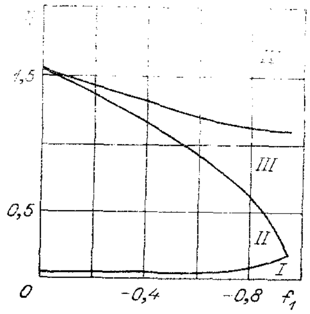


Fig. 2

which is the momentum equation expressed in terms of similarity variables. The last equation is obtained by differentiating c with respect to $f_k$ instead of x. Equation (1.1) is finally brought to the universal form without the explicit presence of the external flow velocity U

$$\frac{\partial\zeta}{d\eta} + (1 - f_1 H)\,\Phi\,\frac{\partial^2\Phi}{\partial\eta^2} + f_1\left[1 - \left(\frac{\partial\Phi}{\partial\eta}\right)^2\right] = \sum_{k=1}^{\infty}\theta_k\left(\frac{\partial\Phi}{\partial\eta}\,\frac{\partial^2\Phi}{\partial\eta\,\partial f_k} - \frac{\partial\Phi}{\partial f_k}\,\frac{\partial^2\Phi}{\partial\eta^2}\right). \quad (1.4)$$

2. Solution to the Universal Equation (the First Stage of the Method). The well-known semiempirical equations of Prandtl (with Van Driest correction) and Caluser are used to determine the shear stress. The expressions for $\tau$ in the inner $\zeta_-$ and outer $\zeta_+$ regions in terms of similarity variables are given by

$$\zeta_- = \frac{1}{\mathrm{Re}^{**}c}\,\frac{\partial^2\Phi}{\partial\eta^2} + \frac{\varkappa^2\eta^2}{c}\left[1 - \exp\left(-\frac{\mathrm{Re}^{**}\sqrt{c}}{A}\,\eta\sqrt{1 + f_1 - f_2\eta}\right)\right]^2\left(\frac{\partial^2\Phi}{\partial\eta^2}\right)^2,$$
$$\zeta_+ = \alpha\,\frac{H}{c}\,\frac{\partial^2\Phi}{\partial\eta^2}, \quad (2.1)$$

where A = 26; $\alpha$ = 0.0168. As indicated in a number of studies, the value of Karman mixing length $\varkappa$ increases from 0.4 for a flat plate to ≈0.6 for larger positive pressure gradients. Since the value of the parameter $f_1$ is −1 at separation, we use the following linear dependence of $\varkappa$ on $f_1$: $\varkappa = 0.41 - 0.1 f_1$ (the Reynolds number dependence of $\varkappa$ is neglected).

The universal equation (1.4) in which the normalized skin friction is determined from Eq. (2.1) was integrated numerically using the marching technique with the locally single parameter approximation. The functions c and H were computed at each step using the velocity profiles obtained at the preceding step:

$$c = \frac{1}{\mathrm{Re}^{**}(1 + f_1)}\,\frac{\partial^2\Phi}{\partial\eta^2}\bigg|_{\eta=0}, \quad H = \int_0^\infty\left(1 - \frac{\partial\Phi}{\partial\eta}\right)d\eta.$$

In order to concentrate computational points close to the wall (in the region where the velocity profile has a large gradient) a logarithmic transformation of the ordinate $\eta$, as suggested in [3], was made. As a result of the solution of the universal equation, a family of velocity profiles shown in Fig. 1 has been obtained ($f_1$ = 0, −0.60, −0.80, −0.95 for the curves 1-4 respectively, and the computations were carried out for $\mathrm{Re}^{**} = 10^4$). Analysis of the computed velocity profiles established the correctness of the "law of 1/2" and the location of the boundary between the characteristic regions of the turbulent boundary layer was determined (Fig. 2): I, viscous sublayer; II, log region; III, region of the "law of 1/2," and IV, outer region. As the separation point is approached the logarithmic region of the velocity profile is gradually displaced by the region of the "the law of 1/2," and the coordinate of the boundary between the inner and outer regions approaches a constant value $\eta$ = 1.1.

3. Solution to the Momentum Equations (the Second Stage of the Method). The family of velocity profiles and the expressions for the universal functions

$$c = 1.32\cdot 10^{-3}/(1 + 0.69 f_1),\ H = 1.30/(1 + 0.22 f_1)\ \text{at}$$
$$-0.7 \leqslant f_1 \leqslant 0,$$

were obtained in the first stage of the similarity solution. It is followed by the second stage in which the ordinary differential equations for the momentum

$$\frac{df_1}{dx} = \frac{\left(\frac{dU}{dx}\right)^{-1}\frac{d^2U}{dx^2}\,f_1 + U^{-1}\frac{dU}{dx}\left[1 - (2 + H)\,f_1\right]}{1 + f_1\,\frac{\partial \ln c}{\partial f_1}},$$

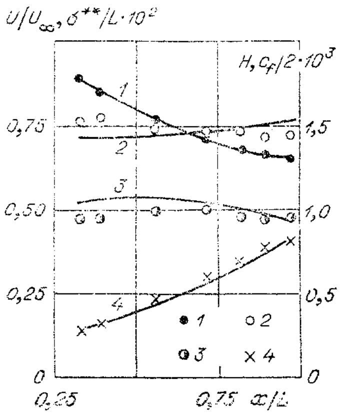


Fig. 3

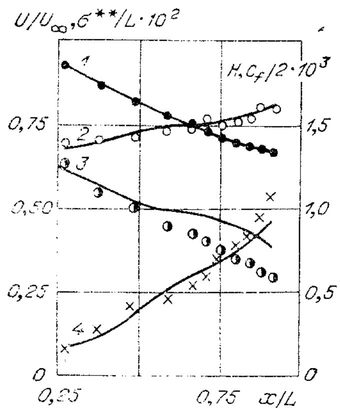


Fig. 4

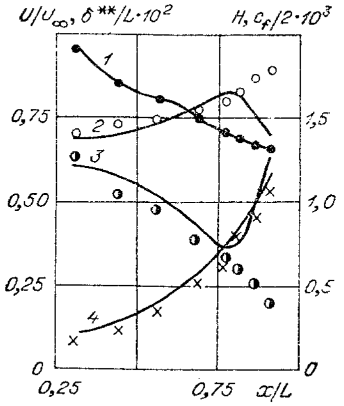


Fig. 5

obtained from Eqs. (1.2) and (1.3) remain to be solved for a particular problem. This equation has been integrated for a number of turbulent boundary layers [4] using the Euler–Cauchy method. Instead of an accurate starting velocity profile an approximate initial condition $f_1 = f_{10}$ was used at $x = x_0$ (the value of $\delta_0^{**}$, partly taking into consideration the history of the flow in the boundary layer, was taken from experimental results). Computational results of three turbulent boundary layers with positive pressure gradients (experiments 2200, 1100, and 1200) are shown in Fig. 3-5. The experiment numbers correspond to the ones given in the proceedings of the Stanford conference [4]. Curves 1 in Fig. 3-5 approximate the experimental distribution of nondimensional velocity at the outer edge of the boundary layer $U/U_\infty$, curves 2-4 are the computed distribution of H, $c_f/2$, $\delta^{**}/L$ ($U_\infty$ is the free stream velocity, L is the reference length of the body), respectively. The experimental data [4] are indicated by points in Fig. 3-5 (1: $U/U_\infty$; 2: H; 3: $c_f/2$; 4: $\delta^{**}/L$).

As is to be expected, the computations based on locally single parameter solution to the universal equation give a very satisfactory agreement with experiment though not for all the standard set of experiments. It is proposed to carry out the solution to the universal equation with single- and two-parameter approximations, which, as in the case of laminar boundary layer, should appreciably improve the agreement between computed and experimental results.

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